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Bivariate Hermite Interpolation and Numerical Curves*

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In this paper, Hermite interpolation by bivariate algebraic polynomials of total degree $\leqslant n$ is considered. The interpolation parameters are the values of a function and its partial derivatives up to some order $n_\nu - 1$ at the nodes $z_\nu = (x_\nu, y_\nu)$, $\nu = 1, ..., s$, where n_ν is the multiplicity of z_ν. The sequence $\mathcal{N} = \{n_1, ..., n_s; n\}$ of multiplicities associated with the degree of interpolating polynomials is investigated. Some results of the paper were announced in [GHS93].

1. Introduction

We define a *scheme* $\mathcal{N} = \{n_1, ..., n_s; n\}$ as a collection of nonnegative integers, where $n_1, ..., n_s$ are the *members*, n is the *degree* and s is the *length* of $\mathcal{N}$. By $\mathcal{S}$ we denote the set of all schemes.

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For a scheme $\mathcal{N} = \{n_1, ..., n_s; n\} \in \mathcal{S}$ we accept that

$$\{n_1, ..., n_s; n\} = \{n_1, ..., n_s, 0, ..., 0; n\}$$

with arbitrary (finite) number of zeros. So dealing with finite number of schemes from $\mathcal{S}$, we may assume that they have the same length or, when it is necessary, that the length of the given scheme is large enough.

We need some notation. For schemes $\mathcal{N} = \{n_1, ..., n_s; n\}$, $\mathcal{M} = \{m_1, ..., m_s; m\}$ the inequality

$\mathcal{N} \leqslant \mathcal{M}$ means that $n \leqslant m$, $n_\nu \leqslant m_\nu$, $\nu = 1, ..., s$ and

$\mathcal{N} + \mathcal{M} := \{n_1 + m_1, ..., n_s + m_s; n + m\}$, $\lambda\mathcal{N} := \{\lambda n_1, ..., \lambda n_s; \lambda n\}$, $\lambda \in Z_+$.

We call $\mathcal{N} \in \mathcal{S}$ an *interpolation* scheme if the following equality holds:

$$\sum_{\nu=1}^{s} \bar{n}_\nu = \overline{n+1}, \tag{1.1}$$

where $\bar{m} = 0 + \cdots + m$.

By $\mathcal{IS}$ we denote the set of all interpolation schemes.

Interpolation schemes with $s = 1$, i.e., the schemes $\{n+1; n\}$, are called *Taylor* schemes.

Definition 1.1. For the interpolation scheme $\mathcal{N} = \{n_1, ..., n_s; n\} \in \mathcal{IS}$ and the node set $\mathcal{Z} = \{z_v = (x_\nu, y_\nu)\}_{\nu=1}^{s} \subset R^2$ the (correct) Hermite interpolation problem $(\mathcal{N}, \mathcal{Z})$ is to find a (unique) polynomial $P \in \pi_n(R^2)$ satisfying conditions

$$\left.\frac{\partial^{i+j} P}{\partial x^i \, \partial y^j}\right|_{z = z_v} = \lambda_{i,j,\nu}, \; i + j < n_\nu, \; v = 1, ..., s, \tag{1.2}$$

for given collection of values

$$\Lambda = \{\lambda_{i,j,\nu}, \; i + j < n_\nu, \; \nu = 1, ..., s\}.$$

In what follows, we briefly express equalities of the form (1.2) by writing:

$$D^{\mathcal{N}} P|_{\mathcal{Z}} = \Lambda.$$

Note that the relation (1.1) means that the number of interpolation conditions in (1.2) is equal to the $\dim \pi_n(R^2)$. We assume that there is no interpolation condition at nodes z_ν with $n_\nu = 0$.

Let us denote by

$$d_{\mathcal{N}}(\mathcal{Z}) := d_{\mathcal{N}}(x_1, y_1, ..., x_s, y_s)$$

the determinant of the system of linear equations (1.2) (with respect to unknown coefficients of P) which consists of the rows:

$$\frac{\partial^{i+j}}{\partial x^i \, \partial y^j} [1, x_\nu, y_\nu, ..., x_\nu^n, x_\nu^{n-1} y_\nu, ..., y_\nu^n]; i+j<n_\nu, \qquad \nu=1, ..., s.$$

Remark 1.2. The following statements are equivalent for any $\mathcal{N} \in \mathcal{I}\mathcal{S}$:

(i) $(\mathcal{N}, \mathcal{Z})$ is not correct, i.e. $d_{\mathcal{N}}(\mathcal{Z})=0$;

(ii) there exists a polynomial P such that

$$P \in \pi_n(R^2),\ P \neq 0,\ D^{\mathcal{N}} P|_{\mathcal{Z}} = 0. \tag{1.3}$$

Since $d_{\mathcal{N}}(\mathcal{Z})$ is a polynomial in variables $x_1, y_1, ..., x_s, y_s$, the correctness of the problem $(\mathcal{N}, \mathcal{Z})$ for some $\mathcal{Z}$ implies that it is correct for almost all $\mathcal{Z} \in R^{2s}$ (with respect to the Lebesgue measure in R^{2s}).

Remark 1.3. The following statements are equivalent for any $\mathcal{N} = \{n_1, ..., n_s; n\} \in \mathcal{S}$:

(i) $n_\nu \leqslant n$ for $\nu=1, ..., s$;

(ii) there exists a node set $\mathcal{Z} \in R^{2s}$ and a polynomial P such that (1.3) holds.

Indeed, if (i) does not hold, i.e. $n_{\nu 0} \geqslant n+1$ for some ν_0 then all the partial derivatives of P up to orde n vanish at the node $z_{\nu 0}$ and hence (ii) does not hold either. On the other hand if (i) holds then taking the nodes $z_1, ..., z_s$ on an arbitrary line $ax+by+c=0$, $|a|+|b|>0$, we get (1.3) setting $P(x, y)=(ax+by+c)^n$.

The Remarks 1.2 and 1.3 imply that the Taylor schemes $\{n+1, n\}$ are the only interpolation schemes which are correct for arbitrary node set (see [LL84] and [JS91] for more general results).

Definition 1.4. We say that the interpolation scheme $\mathcal{N}$ is

(i) regular if $(\mathcal{N}, \mathcal{Z})$ is correct for at least one node set $\mathcal{Z}$,

(ii) singular if $(\mathcal{N}, \mathcal{Z})$ is not correct for any node set $\mathcal{Z}$.

The problem of the full description of regular and singular interpolation schemes still remains open.

Note that in view of Remark 1.2, this problem can be formulated in the following more general way (which enables us to remove the restriction that te scheme $\mathcal{N}$ is an interpolation scheme):

For the given scheme $\mathcal{N} \in \mathcal{S}$ to determine whether for an *arbitrary* $\mathcal{Z}$ there is a polynomial satisfying conditions (1.3). In this case $\mathcal{N}$ is called *singular*. Otherwise it is called *regular*.

Geometrically, the singularity of $\mathcal{N}$ means that for nay node set $\mathcal{Z}$ there exists an algebraic curve of degree $\leqslant n$, passing through $\mathcal{Z}$ with multiplicity $\mathcal{N}$ (i.e. passing through z_ν with multiplicity n_ν, $\nu = 1, ..., s$).

In what follows we will consider mainly this wider problem of singularity and regularity of general schemes.

Let us consider the following "less conditions" class of schemes:

$$LC := \left\{ \mathcal{M} = \{m_1, ..., m_s; m\} \subset Z_+ : \sum_{\nu=1}^{s} \bar{m}_\nu < \overline{m+1} \right\}.$$

It is not hard to see that $\mathcal{M} \in LC$ is singular. Indeed, for any node set $\mathcal{Z}$ finding a polynomial $P_{\mathcal{M}}$ satisfying (1.3) for $\mathcal{M}$ reduces to solving a system of $\sum_{\nu=1}^{s} \bar{m}_\nu$ homogeneous linear equations in $\overline{m+1}$ unknowns.

DEFINITION 1.5. The scheme $\mathcal{N}$ is called a numerical curve if there is a set $M \subset LC$ such that

$$\mathcal{N} = \sum_{\mathcal{M} \in M} \mathcal{M}.$$

We denote the set of numerical curves by NC.

Numerical curves are singular schemes too. Indeed, for any node set $\mathcal{Z}$ the polynomial

$$P := \prod_{\mathcal{M} \in M} P_{\mathcal{M}}$$

satisfies the conditions (1.3).

Conjecture 1.6. [GHS90, P92]. Each singular (interpolation) scheme is a numerical curve.

We have proved in [GHS92] that this conjecture is true under the restriction that there are at most 9 knots with multiplicities $\geqslant 2$.

1.a. *Quadratic Transformation and Reduction of Schemes, Basic Schemes*

DEFINITION 1.7. Let $\mathcal{N} = \{n_1, ..., n_s; n\} \in \mathcal{S}$.

(i) If $n_1 + n_2 \geqslant n + 1$, $n_1 \leqslant n$, $n_2 \leqslant n$, then the reduction of $\mathcal{N}$ with respect to the first two members is the scheme

$$\mathcal{N}^{\times} = \mathcal{N}^{\times}_{1,2} = \{n - n_2, n - n_1, n_3, ..., n_s; 2n - n_1 - n_2\}.$$

(ii) If

$$n_i + n_j \leqslant n, \ 1 \leqslant i < j \leqslant 3, \tag{1.4}$$

then the quadratic transformation of the scheme $\mathcal{N}$ with respect to the first three members is the scheme (cf. [W50], chapter 3, Theorem 7.2)

$$\begin{aligned}\mathcal{N}^* &= \mathcal{N}^*_{1,2,3}\\ &= \{n-n_2-n_3, n-n_1-n_3, n-n_1-n_2, n_4, ..., n_s; 2n-n_1-n_2-n_3\}.\end{aligned}$$

It is not hard to check that

$$\mathcal{N}^{\times} = \{n_1-r, n_2-r, n_3, ..., n_s; n-r\}.$$

$$\mathcal{N}^* = \{n_1-t, n_2-t, n_3-t, n_4, ..., n_s; n-t\},$$

with $r=n_1+n_2-n$, $t=n_1+n_2+n_3-n$.

We define reduction or quadratic transformation with respect to the other members in the similar way.

Remark 1.8. If the condition (1.4) holds then it holds also for $\mathcal{N}^*$, i.e. $n_i^*+n_j^* \leqslant n^*$, $1 \leqslant i<j \leqslant 3$. This means that the same quadratic transformation can be applied once more. Moreover $(\mathcal{N}^*)^*=\mathcal{N}$.

The following theorem for interpolation schemes was proved in [GHS92]. The proof in the general case essentialy is the same.

THEOREM 1.9. *Let* $\mathcal{N}=\{n_1, ..., n_s; n\} \in \mathcal{S}$. *Then*

(i) *if* $n_1+n_2=n+r \geqslant n+1$, $n_1 \leqslant n$, $n_2 \leqslant n$, *then the schemes* $\mathcal{N}$ *and* $\mathcal{N}^{\times}$ *are both simultaneously singular or not,*

(ii) *if* $n_i+n_j \leqslant n$, $1 \leqslant i<j \leqslant 3$, *then the schemes* $\mathcal{N}$ *and* $\mathcal{N}^*$ *are both simultaneously singular or not.*

This theorem reduces the investigation of an arbitrary scheme to the following two cases:

(1) $n_{\nu_0} \geqslant n+1$ for some ν_0,

(2) $n_i+n_j+n_k \leqslant n$, $1 \leqslant i<j<k \leqslant s$.

In the first case the scheme obviously is regular (Remark 1.3).

DEFINITION 1.10. A scheme $\mathcal{N}=\{n_1, ..., n_s; n\} \in \mathcal{S}$ satisfying the above condition (2) is called basic.

Let *BS* be the class of all basic schemes.

Conjecture 1.11. [GHS90]. A basic scheme is singular if and only if it belongs to *LC*.

We have proved in [GHS92] that Conjecture 1.11 is true in the case

$$\sum_{\nu:\, n_\nu > 1} n_\nu \leqslant 3n$$

and the above two conjectures are equivalent for interpolation schemes. Here we get, in particular, the equivalence in the general case (see Corollary 5.11).

2. Equivalence of Schemes, Prime Curves

Let us define the *intersection product* of schemes (numerical curves) as follows:

$$\langle \mathcal{N}, \mathcal{M} \rangle := \sum_{\nu=1}^{s} n_\nu m_\nu - nm,$$

and set

$$\langle \mathcal{N} \rangle := \langle \mathcal{N}, \mathcal{N} \rangle = \sum_{\nu=1}^{s} n_\nu^2 - n^2, \qquad [\mathcal{N}] := \sum_{\nu=1}^{s} n_\nu - 3n.$$

Denote by $\bar{\mathcal{N}}$ the difference of the number of conditions of $\mathcal{N}$ and $(\dim \pi_n(R^2) - 1)$, i.e.

$$\bar{\mathcal{N}} := \sum_{\nu=1}^{s} \bar{n}_\nu - \overline{n+1} + 1.$$

Thus we have $\bar{\mathcal{N}} = 1$ for interpolation schemes and $\bar{\mathcal{N}} \leqslant 0$ is equivalent to $\mathcal{N} \in LC$. Note that

$$\bar{\mathcal{N}} = (\langle \mathcal{N} \rangle + [\mathcal{N}])/2.$$

It is not hard to check the following properties of quadratic transformation (see [GHS92b]) and reduction:

Lemma 2.1. *Let $\mathcal{N}, \mathcal{M} \in \mathcal{S}$, then*

(i) $(\mathcal{N}^*)^* = \mathcal{N}$,

(ii) $\langle \mathcal{N}, \mathcal{M} \rangle = \langle \mathcal{N}^*, \mathcal{M}^* \rangle$,

(iii) $[\mathcal{N}] = [\mathcal{N}^*]$, $\langle \mathcal{N} \rangle = \langle \mathcal{N}^* \rangle$, $\bar{\mathcal{N}} = \bar{\mathcal{N}}^*$,

(iv) $[\mathcal{N}] = [\mathcal{N}^\times] - r$, $\langle \mathcal{N} \rangle = \langle \mathcal{N}^\times \rangle + r^2$, $\bar{\mathcal{N}} = \bar{\mathcal{N}}^\times + \overline{r-1}$.

Definition 2.2. (i) The schemes $\mathcal{N}, \mathcal{M}$ are called (quadratically) equivalent ($\mathcal{N} \sim \mathcal{M}$), if one of them can be obtained from the other by means of the following operations: quadratic transformation, rearrangement of the members, adding zero members.

(ii) We say that the scheme $\mathcal{N}$ reduces to the scheme $\mathcal{M}$ ($\mathcal{N} \to \mathcal{M}$), if $\mathcal{M}$ can be obtained from $\mathcal{N}$ by means of reduction and (or) above mentioned operations of equivalence.

It follows from Theorem 1.9 that

Remark 2.3. If the scheme $\mathcal{N}$ reduces to the scheme $\mathcal{M}$, then they are either simultaneously singular or simultaneously regular.

Now we give the definition of prime numerical curves which play an essential role in our investigation:

Those schemes which are equivalent to $\{1, 1; 1\}$ are called prime curves. We denote the class of prime curves by PC:

$$PC := \{\mathcal{N} \in \mathcal{S}: \mathcal{N} \sim \{1, 1 : 1\}\}.$$

Using the Lemma 2.1 we get the following

COROLLARY 2.4. *Let* $\mathcal{N}, \mathcal{M} \in \mathcal{S}$.

(ii) *If* $\mathcal{N} \sim \mathcal{M}$ *then*

$$[\mathcal{N}] = [\mathcal{M}], \langle \mathcal{N} \rangle = \langle \mathcal{M} \rangle, \bar{\mathcal{N}} = \bar{\mathcal{M}}.$$

(ii) *If* $\mathcal{N} \to \mathcal{M}$ *then*

$$[\mathcal{N}] \leqslant [\mathcal{M}], \langle \mathcal{N} \rangle \geqslant \langle \mathcal{M} \rangle, \bar{\mathcal{N}} \geqslant \bar{\mathcal{M}}.$$

The first two inequalities are strict if a reduction was used to obtain $\mathcal{M}$ *from* $\mathcal{N}$ *and the third one is strict if and only if a reduction with* $r > 1$ *was used.*

In particular, for $A \in PC$, we have

$$[\mathcal{A}] = -1, \langle \mathcal{A} \rangle = 1, \bar{\mathcal{A}} = 0. \tag{2.1}$$

Next we are going to develop an essential tool—to prove that a quadratic transformation can be applied to any three members of a prime curve different from $\{1, 1; 1\}$. This is expressed in the following

THEOREM 2.5. *Suppose* $\mathcal{A} = \{\alpha_1, ..., \alpha_s; \alpha\} \in PC$, $\mathcal{A} \neq \{1, 1; 1\}$. *Then*

$$\max_{1 \leqslant i \leqslant j \leqslant s} (\alpha_i + \alpha_j) \leqslant \alpha.$$

In order to prove this we need several lemmas.

LEMMA 2.6 [GHS92b]. *If* $\bar{\mathcal{N}} \leqslant 0$ *and* $\langle \mathcal{N} \rangle \geqslant 1$ *then*

$$\max_{1 \leqslant i<j<k \leqslant s} (n_i + n_j + n_k) > n. \tag{2.2}$$

LEMMA 2.7. *Suppose* $\mathcal{N} = \{n_1, n_2, n_3, n_4; n\}$ *with*

$$\bar{\mathcal{N}} \leqslant 0, \langle \mathcal{N} \rangle \geqslant 1, n_4 = 0, 1.$$

Then

$$\max_{1 \leqslant i<j \leqslant 4} (n_i + n_j) > n. \tag{2.3}$$

Proof. Let $\mathcal{N} = \{n_1, n_2, n_3, 1; n\}$ satisfies the conditions of the Lemma and (2.3) does not hold. Consider the scheme $\mathcal{N}^* = \{n_1^*, n_2^*, n_3^*, 1; n^*\}$. According to Lemma 2.1 iii) both $\mathcal{N}$ and $\mathcal{N}^*$ satisfy the condition of Lemma 2.6. Therefore we have $n_1 + n_2 + n_3 \geqslant n$ and $n_1^* + n_2^* + n_3^* \geqslant n^*$ which imply $n_1 + n_2 + n_3 = n$. This, in view of (2.2) means that one of the members n_1, n_2, n_3, say n_3, equals to 0 and hence $n_1 + n_2 = n$. Now excluding the case $\{n, 0, 0, 1; n\}$, i.e. the case $n_1 n_2 = 0$ we have

$$\langle \mathcal{N} \rangle = n_1^2 + n_2^2 + 1 - (n_1 + n_2)^2 < 0,$$

which contradicts the assumption. The proof in the case $n_4 = 0$ is similar (and simpler).

LEMMA 2.8. *Let* $\mathcal{N}, \mathcal{M} \in \mathcal{S}$, $\langle \mathcal{N} \rangle \leqslant 1$, $\langle \mathcal{M} \rangle \leqslant 1$ *and* $n_\nu = m_\nu = \delta \geqslant 1$ *for some* $\nu = 1, ..., s$. *Then*

$$\langle \mathcal{N}, \mathcal{M} \rangle \leqslant 1, \tag{2.4}$$

the equality being possible in the following three cases only:

(a) $\mathcal{N} = \mathcal{M}$ *and* $\langle \mathcal{N} \rangle = \langle \mathcal{M} \rangle = 1$,

(b) $\langle \mathcal{N} \rangle = \langle \mathcal{M} \rangle = \delta = 1$ *and*

$$\{m_1, ..., m_{\nu-1}, m_{\nu+1}, ..., m_s; m\} = \lambda \{n_1, ..., n_{\nu-1}, n_{\nu+1}, ..., n_s; n\}, \tag{2.5}$$

(c) $\delta = 1$ *and one of schemes in* (2.5) (*without* λ) *identically equals to zero.*

Proof. To prove (2.4) we can assume, without loss of generality, that $\langle \mathcal{N} \rangle = \langle \mathcal{M} \rangle = 1$, (we can achieve this by adding one or zero members to the schemes $\mathcal{N}$ and $\mathcal{M}$ without changing $\langle \mathcal{N}, \mathcal{M} \rangle$).

Let $v = s$. We use the following familiar identity:

$$\sum_{v=1}^{s} n_v^2 \sum_{v=1}^{s} m_v^2 = \left(\sum_{v=1}^{s} n_v, m_v \right)^2 + \sum_{1 \leqslant v < \mu \leqslant s} (n_v m_\mu - n_\mu m_v)^2.$$

Retaining the summands with $\mu = s$ in the last sum, we obtain

$$(n^2+1)(m^2+1) \geqslant \left(\sum_{v=1}^{s} n_v m_v \right)^2 + \delta^2 \sum_{v=1}^{s} (n_v - m_v)^2 \tag{2.6}$$

Since $\delta \geqslant 1$, we get

$$n^2 m^2 \geqslant \left(\sum_{v=1}^{s} n_v m_v - 1 \right)^2.$$

Therefore

$$\sum_{v=1}^{s} n_v m_v \leqslant nm + 1.$$

Now, if equality holds here in the case $\langle \mathcal{N} \rangle = \langle \mathcal{M} \rangle = 1$, then we have equalities in (2.6) with δ replaced by 1. And this implies either the case (a) or $\delta = 1$ and

$$\sum_{1 \leqslant v < \mu < s} (n_v m_\mu - n_\mu m_v)^2 = 0,$$

which completes the proof in this case. Now let us have the equality in the case $\langle \mathcal{N} \rangle \leqslant 1$, $\langle \mathcal{M} \rangle \leqslant 1$, with at least one of this inequalities being strict. Then we get equalities here again by adding one and zero members to $\mathcal{N}$ and $\mathcal{M}$ without changing $\langle \mathcal{N}, \mathcal{M} \rangle$. Now, as in the previous case, we have equality (2.5) for the resulting schemes, (the case (a) here is excluded) and hence λ equals to zero.

LEMMA 2.9. *For* $\mathcal{N}, \mathcal{M} \in \mathcal{S}$, *suppose that* $\langle \mathcal{N} \rangle \leqslant 1$, $\langle \mathcal{M} \rangle \leqslant 1$, $\langle \mathcal{N}, \mathcal{M} \rangle \geqslant 1$, *and* $\deg \mathcal{N} \geqslant \deg \mathcal{M}$. *Then*

$$n_v > m_v \qquad \textit{for all} \quad v = 1, ..., s \tag{2.7}$$

Proof. Assume that (2.7) does not hold for $v = 1$ and let

$$n_1 = a, m_1 = a + b, a \geqslant 0, b > 0.$$

Consider the schemes

$$\mathcal{N}' = \{a + b, n_2, ..., n_s; n\},$$

$$\mathcal{M}' = \{a, m_2, ..., m_s; m\}.$$

Then

$$\langle \mathcal{N}' \rangle \leqslant 1 + M, \langle \mathcal{M}' \rangle \leqslant 1 - M,$$

with $M = 2ab + b^2 \geqslant 1$ and we get

$$(nm+1)^2 \leqslant \left(\sum_{v=1}^{s} n_v m_v \right)^2 = \left(\sum_{v=1}^{s} n'_v m'_v \right)^2$$

$$\leqslant (n^2 + 1 + M)(m^2 + 1 - M).$$

Therefore

$$M(n-m)(n+m) + M^2 \leqslant (n-m)^2,$$

which obviously is a contradiction.

Now we readily get from Lemmas 2.8 and 2.9

Remark 2.10. Let the conditions of Lemma 2.9 hold. Then equality of one of the nonzero members in (2.7) implies (2.5) with

$$\langle \mathcal{N}, \mathcal{M} \rangle = 1,$$

and equalities of two of the nonzero members in (2.7), as well as the equality of degrees, implies $\mathcal{N} = \mathcal{M}$.

Proof of Theorem 2.5. Suppose $A = \{\alpha_1, ..., \alpha_s; \alpha\} \in PC$, $A \neq \{1, 1; 1\}$ and

$$\alpha_p + \alpha_q \geqslant \alpha + 1 \tag{2.8}$$

for some p, q. Denote

$$E = E_{p,q} := \{e_1, ..., e_s; e\},$$

with $e_p = e_q = e = 1$ and $e_v = 0$ if $v \neq p, q$. So we have

$$\langle A, E \rangle \geqslant 1.$$

Assume that

$$A = T_1 \cdots T_m F, \tag{2.9}$$

where $F = E_{k,l}$ for some k, l and T_i is a quadratic transformation with respect to some triplet of members.

Suppose that m is the minimal number for which there exists an F(i.e. k, l) and $A \neq \{1, 1; 1\}$ satisfying (2.9) and (2.8) for some p, q.

Denote for $i = 1, ..., m$

$$A_i := T_i \cdots T_1 A = T_{i+1} \cdots T_m F, \; B_i := T_i \cdots T_1 E, \tag{2.10}$$

with $A_0 = A$, $A_m = F$, $B_0 = E$. Note that $T_i \cdots T_1$ is applicable to E because of the minimality of m and the same is true for the other transformations in (2.10) in view of Remark 1.8. Of course all of the schemes A_i, B_i belong to PC. In view of Lemma 2.1(ii) we have

$$\langle A, E \rangle = \langle A_i, B_i \rangle \geqslant 1, i = 1, ..., m. \tag{2.11}$$

Now, if there are two (nonzero) equal corresponding members of A_i, B_i or deg $A_i = \deg B_i$, for some i, then Remark 2.10 implies $A_i = B_i$ and we get from (2.10)

$$A = T_1 \cdots T_i A_i = T_1 \cdots T_i B_i = E = \{1, 1; 1\},$$

which contradicts the minimality of m. On the other hand, it is easy to check, that

$$\deg A_0 > \deg B_0 = 1 \qquad \text{and} \qquad 1 = \deg A_m < \deg B_m.$$

Hence there is a v, $0 \leqslant v \leqslant m-1$, such that

$$\deg A_v > \deg B_v, \quad \deg A_{v+1} < \deg B_{v+1},$$

besides we have

$$A_{v+1} = T_{v+1} A_v, B_{v+1} = T_{v+1} B_v.$$

Assume that T_{v+1} acts on some triplet. Then in view of (2.1), (2.11), Lemma 2.9 and Remark 2.10, the prime curves A_v, $B_v (A_v \neq B_v)$ can have at most one nonzero member positioned outside the triplet. This member (if there is one) necessarily equals 1 by Lemma 2.8.

Hence A_v, B_v are of the form of the schemes in Lemma 2.7 and this again contradicts the minimality of m.

3. The Classes of Schemes BS^* and $BS^{\times *}$

Definition 3.1. (i) A scheme which is equivalent to some basic scheme (see Definition 1.10) is called an *e*-basic scheme.

(ii) A scheme which can be reduced to some basic scheme is called an *r*-basic scheme.

We denote by BS^* and $BS^{\times *}$ the classes of *e*-basic and *r*-basic schemes respectively. Of course we have

$$BS \subset BS^* \subset BS^{\times *}.$$

Note that Theorem 1.9 and Remark 1.3. imply

Remark 3.2. *If* $\mathcal{N}$ is singular then $\mathcal{N} \in BS^{\times *}$.

In particular, the following theorem shows, that the quadratic transformation can be applied to any triplet of members of *e*-basic scheme (setting $A = E_{i,j}$ in (ii) below).

THEOREM 3.3. *If* A, B *are prime curves,* $A \neq B$, $\mathcal{N}$ *is a basic scheme with decreasing members and* $\mathcal{M}$ *is e-basic, then*

(i) $\langle A, B \rangle \leqslant 0$,

(ii) $\langle A, \mathcal{N} \rangle \leqslant n_1 + n_2 - n$,

(iii) $\langle A, \mathcal{M} \rangle \leqslant 0$.

Proof. (i) Let $A = T_1 \cdots T_m E_{1,2}$, where T_i is a quadratic transformation. If $T_m \cdots T_1$ is applicable to B, then

$$\langle A, B \rangle = \langle E_{1,2}, T_m \cdots T_1 B \rangle,$$

and it remains to apply Theorem 2.5. We will come to the same situation if some T_i can not be applied to $T_{i-1} \cdots T_1 B$, since then the latter scheme must be $\{1, 1; 1\}$ (see Theorem 2.5).

(ii) We use induction on deg A. The case deg $A = 1$ is obvious. We can assume that the members of both of A and $\mathcal{N}$ are in decreasing order, since $\langle A, \mathcal{N} \rangle$ is maximal in this case. Now we have

$$\langle A, \mathcal{N} \rangle = \langle A^*, \mathcal{N}^* \rangle,$$

and in view of Lemma 2.6 $\deg A^* < \deg A$.

If $A^* = \{1, 1; 1\}$ then $A = \{1, 1, 1, 1, 1, 2\}$ and

$$\langle A, \mathcal{N} \rangle = n_1 + n_2 + n_3 + n_4 + n_5 - 2n \leqslant n_1 + n_2 - n.$$

Otherwise, let us rearrange the members of A^* in decreasing order and denote it by A_0. Since $\mathcal{N}^*$ automatically maintains the decreasing order, we will have

$$\langle A^*, \mathcal{N}^* \rangle \leqslant \langle A_0, \mathcal{N}^* \rangle = \langle A_0^*, \mathcal{N} \rangle,$$

with

$$\deg A_0^* < \deg A_0 = \deg A^*.$$

It remains to use the induction hypothesis.

(iii) Let $\mathcal{M} = T_1 \cdots T_m \mathcal{N}$, $\mathcal{N} \in BS$. Suppose

$$\langle A, \mathcal{M} \rangle \geqslant 1,$$

for some $A \in PC$. It is not hard to see that T_i is applicable to the scheme $T_{i-1} \cdots T_1 A \in PC$ for each $i = 1, ..., m$. Indeed, if the scheme differs from $\{1, 1; 1\}$ then we get this from Theorem 2.5. If the scheme equals some $E_{k,l}$ then

$$\langle E_{k,l}, T_i \cdots T_1 \mathcal{M} \rangle \geqslant 1,$$

and the triplet of T_i cannot include (i, j) (see Remark 1.8), hence again T_i can be applied. Now Lemma 2.1(ii) implies

$$\langle T_m \cdots T_1 A, \mathcal{N} \rangle \geqslant 1,$$

which contradicts (ii).

The following Theorem gives a characterization of the class BS^*

THEOREM 3.4. (i) *The scheme $\mathcal{N}$ is e-basic if and only if*

$$\langle A, \mathcal{N} \rangle \leqslant 0 \qquad \text{for all} \quad A \in PC \tag{3.1}$$

(ii) *If $\mathcal{N}_\nu \in BS^*$, $\nu = 1, ..., k$, then*

$$\sum_{\nu=1}^{k} \lambda_\nu \mathcal{N}_\nu \in BS^*,$$

where $\lambda_\nu \in Z_+$.

Proof. Of course (i) implies (ii). In order to prove (i), it is enough to show that (3.1) implies $\mathcal{N} \in BS^*$ (see Theorem 3.3(iii)). We will prove this using induction on $\deg \mathcal{N}$. The case $\deg \mathcal{N} = 0$ is obvious, since then (3.1) implies $\mathcal{N} = 0$.

If $\deg \mathcal{N} > 0$ and $\mathcal{N} \notin BS$, then there are distinct i, j, k with $n_i + n_j + n_k > n$. According to (3.1) the quadratic transformation T with this triplet is applicable to $\mathcal{N}$, moreover, $\deg T\mathcal{N} < \deg \mathcal{N}$. Let $A \in PC$. If $A = E_{l,m}$ with (l, m) inside the triplet, then Remark 1.8 implies

$$\langle A, T\mathcal{N} \rangle \leqslant 0.$$

Otherwise $TA \in PC$ and using again Remark 1.8 and Lemma 2.1(ii), we get

$$\langle A, T\mathcal{N} \rangle = \langle TA, \mathcal{N} \rangle \leqslant 0.$$

Now on the basis of induction hypothesis we conclude that $T\mathcal{N}$ and therefore $\mathcal{N}$ are e-basic schemes.

4. The Canonical Decomposition of Schemes

Now we are in a position to present the main result of this paper—the *canonical decomposition* of r-basic schemes.

THEOREM 4.1. *Let $\mathcal{N} \in BS^{\times *}$. Then there exist a finite set of prime curves $PC_{\mathcal{N}}$, an e-basic scheme $\mathcal{N}^{\downarrow}$ and natural numbers $\mu_A = \mu_{A,\mathcal{N}}$, $(A \in PC_{\mathcal{N}})$, such that*

$$\mathcal{N} = \sum_{A \in PC_{\mathcal{N}}} \mu_A A + \mathcal{N}^{\downarrow}, \tag{4.1}$$

with the following orthogonality conditions:

$$\begin{aligned} \langle A, B \rangle = 0 \qquad &\textit{for all} \quad A, B \in PC_{\mathcal{N}},\ A \neq B, \\ \langle A, \mathcal{N}^{\downarrow} \rangle = 0 \qquad &\textit{for all} \quad A \in PC_{\mathcal{N}}. \end{aligned} \tag{4.2}$$

Moreover, the decomposition (4.1) *with conditions* (4.2) *is unique.*

Remark 4.2. Suppose

$$\mathcal{N}_0 = T_1 \cdots T_m \mathcal{N}, \tag{4.3}$$

where $\mathcal{N}_0$ is e-basic and T_i is a quadratic transformation or reduction. Then $\mathcal{N}^{\downarrow} \sim \mathcal{N}_0$, more pricisely we have

$$\mathcal{N}^{\downarrow} = T_{i_k} \cdots T_{i_1} \mathcal{N}_0, \tag{4.4}$$

where $T_{i_1}, ..., T_{i_k}$, $i_1 < \cdots < i_k$, are all the quadratic transformations from $T_1, ..., T_m$.

We will prove Theorem 4.1 and Remark 4.2 together.

Proof. First we will prove the Remark and the existence part of the Theorem. Suppose the scheme $\mathcal{N}_0$ is defined as in (4.3). We will use induction on m. If $m = 0$, i.e. $\mathcal{N} \in BS^*$, the decomposition is trivial: $\mathcal{N} = \mathcal{N}$. For $m > 0$ the following two cases are possible:

(a) T_m is a quadratic transformation,

(b) T_m is a reduction.

In both cases we apply the induction hypothesis to the scheme

$$\mathcal{N}_1 = \{n_1^1, ..., n_s^1; n^1\} = T_m \mathcal{N}.$$

Since

$$\mathcal{N}_0 = T_1 \cdots T_{m-1} \mathcal{N}_1, \tag{4.5}$$

we get from (4.4), that $T_{i_k} = T_m$ in case (a) and this implies the decomposition

$$\mathcal{N}_1 = T_m \mathcal{N} = \sum_{A \in PC_{\mathcal{N}_1}} \mu_A A + \mathcal{N}_1^{\downarrow}, \tag{4.6}$$

with

$$\mathcal{N}_1^{\downarrow} = T_{i_{k-1}} \cdots T_{i_1} \mathcal{N}_0.$$

Now we are going to apply the transformation T_m to the both sides of (4.6), i.e. to all of the schemes appearing there. Let us justify this application. Of course T_m is applicable to $\mathcal{N}_1 = T_m \mathcal{N}$ (see Remark 1.8). Next T_m is applicable to all $A \in PC_{\mathcal{N}_1}$ with $\deg A > 1$ and to $\mathcal{N}_1^{\downarrow} \in BS^*$ due to Theorems 2.5 and 3.3(iii) respectively.

If $E = E_{i,j} \in PC_{\mathcal{N}_1}$, then using the orthogonality conditions for (4.6) we readily get $n_i^1 + n_j^1 = n^1 + \mu_E$. Now the Remark 1.8 implies that the triplet of the transformation T_m cannot include (i, j). And hence T_m can be applied to E.

So applying T_m to (4.6) we get

$$\mathcal{N} = \sum_{A \in PC_{\mathcal{N}_1}} \mu_A T_m A + \mathcal{N}^{\downarrow},$$

with (4.4).

The orthogonality conditions can be easily checked using the orthogonality of decomposition (4.6) and Lemma 2.1(ii).

Let us consider now the case (b). In this case we have (4.5) and $i_k \leqslant m-1$. Hence, using the induction hypothesis we get the decomposition:

$$\mathcal{N}_1 = T_m \mathcal{N} = \sum_{A \in PC_{\mathcal{N}_1}} \mu_A A + \mathcal{N}^{\downarrow},$$

with (4.4). Assume the reduction T_m acts on the pair (i, j). Then

$$\mathcal{N}_1 = T_m \mathcal{N} = \mathcal{N} - (n_i + n_j - n)E, \; (E = E_{i,j})$$

and we get the following canonical decomposition

$$\mathcal{N} = (n_i + n_j - n)E + \sum_{A \in PC_{\mathcal{N}_1}} \mu_A A + \mathcal{N}^{\downarrow} \tag{4.7}$$

Let us check the orthogonality properties. Taking the product of E with both sides of (4.7), we obtain

$$0 = \sum_{A \in PC_{\mathcal{N}_1}} \mu_A \langle E, A \rangle + \langle E, \mathcal{N}^{\downarrow} \rangle.$$

This equality and the orthogonality conditions of the decomposition of $\mathcal{N}_1$ imply that $E \notin PC_{\mathcal{N}_1}$ since otherwise we get μ_E in the right hand side of the above equality. On the other hand note that according to Theorems 2.5, 3.3(iii) we have

$$\langle E, A \rangle \leqslant 0, \qquad \text{for all} \quad A \in PC_{\mathcal{N}_1}$$

and

$$\langle E, \mathcal{N}^{\downarrow} \rangle \leqslant 0.$$

And hence all of the above products are equal to zero.

The uniqueness part of Theorem easily follows from the following

Remark 4.3. A prime scheme A belongs to $PC_{\mathcal{N}}$ in the canonical decomposition (4.1) if and only if $\langle A, \mathcal{N} \rangle > 0$, moreover, we have $\langle A, \mathcal{N} \rangle = \mu_A$ in that case.

This readily follows from the orthogonality conditions, in one direction, and Theorem 3.3(i), (iii) in the other. The proof of Theorem 4.1 is complete.

COROLLARY 4.4. *For any $\mathcal{N} \in BS^{\times *}$ there are only a finite number ($\leqslant \deg \mathcal{N}$) of prime schemes A with $\langle A, \mathcal{N} \rangle > 0$ and any two such schemes are orthogonal. Moreover we have $\mathcal{N} \geqslant \mathcal{M}$, where $\mathcal{M}$ is the sum of all such prime schemes.*

COROLLARY 4.5. *The following relations hold for the canonical decomposition* (4.1) *of any scheme $\mathcal{N} \in BS^{\times *}$:*

$$\text{(i)} \qquad [\mathcal{N}] = -\sum_{A \in PC_{\mathcal{N}}} \mu_A + [\mathcal{N}^{\downarrow}]$$

$$\text{(ii)} \qquad \langle \mathcal{N} \rangle = \sum_{A \in PC_{\mathcal{N}}} \mu_A^2 + \langle \mathcal{N}^{\downarrow} \rangle,$$

$$\text{(iii)} \qquad \overline{\mathcal{N}} = \sum_{A \in PC_{\mathcal{N}}} \overline{\mu_A - 1} + \overline{\mathcal{N}}^{\downarrow}.$$

COROLLARY 4.6. (i) *Suppose that the scheme $\mathcal{N}$ has a decomposition* (4.1) *with* (4.2). *Then $\mathcal{N} \to \mathcal{N}^{\downarrow}$ which implies $\mathcal{N} \in BS^{\times *}$.*

(ii) *if $\mathcal{N} \to \mathcal{M}_1$ and $\mathcal{N} \to \mathcal{M}_2$, where $\mathcal{M}_1$, $\mathcal{M}_2$ are e-basic schemes, then $\mathcal{M}_1 \sim \mathcal{M}_2$.*

Proof. (i) We use induction on the number m of prime curves in (4.1). The cae $m = 0$ is trivial. Consider now the decomposition (4.1). Applying some quadratic transformations to both sides of (4.1) we can get a canonical decomposition of $N_0 = \{n_1^0, ..., n_s^0; n^0\}$, $(\mathcal{N} \sim \mathcal{N}_0)$, which involves some $E_{i,j}$ as a prime curve. Then the orthogonality conditions imply that

$$n_i^0 + n_j^0 = n^0 + \mu_{E_{i,j}}$$

and we can apply reduction with respect to the members in places i, j to both sides of canonical decomposition of N_0. We will have $(m-1)$ prime curves in the resulting decomposition of $\mathcal{N}_1 := \mathcal{N}_0 - \mu_{E_{i,j}} E_{i,j}$. It remains to use the induction hypothesis and the obvious transitivity of $\rightarrow$.

Part (ii) readily follows from Remark 4.2 and the uniqueness of decomposition (4.1).

The following lemma is needed for the next characterization of r-basic schemes.

LEMMA 4.7. *Let $\mathcal{N} \in BS^{\times *}$ and $\mathcal{N} \rightarrow \mathcal{M}$. Then $\mathcal{M} \in BS^{\times *}$*

Proof. Suppose

$$\mathcal{M} = T_1 \cdots T_m \mathcal{N},$$

where T_i is a quadratic transformation or reduction. Consider the canonical decomposition of $\mathcal{N}$. It is not hard to see, just as in the proof of Theorem 4.1, that $\mathcal{M}$ has a similar decomposition with

$$\mathcal{M}^{\downarrow} = T_{i_k} \cdots T_{i_1} \mathcal{N}^{\downarrow},$$

where $T_{i_1}, ..., T_{i_k}$, $i_1 < \cdots i_k$, are all the quadratic transformations from $T_1, ..., T_m$.

COROLLARY 4.8. *The following conditions are equivalent for any $\mathcal{N} \in \mathcal{S}$:*

(i) $\mathcal{N} \notin BS^{\times *}$,

(ii) *There is a scheme $\mathcal{M} = \{m_1, ..., m_s; m\}$, with $m_i \geqslant m+1$ for some $1 \leqslant i \leqslant s$ such that $\mathcal{N} \rightarrow \mathcal{M}$.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. In order to check (ii) $\Rightarrow$ (i) suppose $\mathcal{M}$ satisfies (ii) while $\mathcal{N} \in BS^{\times *}$. Then by Lemma 4.7 $\mathcal{M} \in BS^{\times *}$. Consider the canonical decomposition of $\mathcal{M}$. We have $\alpha_i \leqslant \alpha$ for all prime curves $A = \{\alpha_1, ..., \alpha_s; \alpha\}$, in this decomposition. Hence $m_{i0}^{\downarrow} \geqslant m^{\downarrow} + 1$ which contradicts Theorem 3.3(iii).

5. THE CANONICAL DECOMPOSITION OF NUMERICAL CURVES

As it was mentioned in Remark 3.2, Theorem 1.9 and Remark 1.3 imply that each singular scheme is r-basic. Here we give a direct proof of the following

THEOREM 5.1. *If $\mathcal{N}$ is a numerical curve then $\mathcal{N} \in BS^{\times *}$ with*

$$\mathcal{N}^{\downarrow} \in LC, \qquad \text{and} \qquad \langle \mathcal{N}^{\downarrow} \rangle \leqslant 0. \tag{5.1}$$

First we need some lemmas. The following Lemma readily follows from Corollary 2.4 and Lemma 2.6.

LEMMA 5.2. *If $\mathcal{N} \in BS^* \cap LC$ then $\langle \mathcal{N} \rangle \leqslant 0$.*

LEMMA 5.3. (i) *If $\langle \mathcal{N} \rangle \leqslant 0$, then $\mathcal{N} \in BS^{\times *}$ with $\langle \mathcal{N}^{\downarrow} \rangle \leqslant 0$:*
(ii) *If $\mathcal{N} \in LC$, then $\mathcal{N} \in BS^{\times *}$ with $\mathcal{N}^{\downarrow} \in LC$ and $\langle \mathcal{N}^{\downarrow} \rangle \leqslant 0$.*

Proof. If $\mathcal{N} \notin BS^{\times *}$ then according to Corollary 4.8 there exists $\mathcal{M} \in \mathcal{S}$ with $\mathcal{N} \to \mathcal{M}$ and $m_i \geqslant m+1$ for some i. This means that $\bar{\mathcal{M}} \geqslant 1$ and $\langle \mathcal{M} \rangle \geqslant 1$. Hence, in view of Corollary 2.4(ii), we have $\bar{\mathcal{N}} \geqslant 1$ and $\langle \mathcal{N} \rangle \geqslant 1$. This ensures that $\mathcal{N} \in BS^{\times *}$ for both cases.
Now recall that $\mathcal{N} \to \mathcal{N}^{\downarrow} \in BS^*$. To end the proof, it remains to use Corollary 2.4(ii) and Lemma 5.2.

Theorems 3.3(i) and 3.4.(i) imply

LEMMA 5.4.

(i) *Let $\mathcal{M} \in BS^*$ and $A, B \in PC$ with $\langle A, B \rangle < 0$. Then $\mathcal{M} + A + B \in BS^*$.*

(ii) *Let $\mathcal{M} \in BS^*$ and $A \in PC$ with $\langle A, \mathcal{M} \rangle = -\mu < 0$. Then $\mathcal{M} + \mu A \in BS^*$.*

The following lemma follows from Cauchy's inequality and the identities:

$$\begin{gathered} \langle \mathcal{N} + \mathcal{M} \rangle = \langle \mathcal{N} \rangle + \langle \mathcal{M} \rangle + 2 \langle \mathcal{N}, \mathcal{M} \rangle, \\ \overline{\mathcal{N} + \mathcal{M}} = \bar{\mathcal{N}} + \bar{\mathcal{M}} + \langle \mathcal{N}, \mathcal{M} \rangle. \end{gathered} \tag{5.2}$$

LEMMA 5.5. (i) *Let $\langle \mathcal{N} \rangle \leqslant 0$, $\langle \mathcal{M} \rangle \leqslant 0$. Then $\langle \mathcal{N}, \mathcal{M} \rangle \leqslant 0$ and $\langle \mathcal{N} + \mathcal{M} \rangle \leqslant 0$.*
(ii) *Let $\langle \mathcal{N} \rangle \leqslant 1$, $\langle \mathcal{M} \rangle \leqslant 1$ and $\langle \mathcal{N}, \mathcal{M} \rangle < 0$. Then $\langle \mathcal{N} + \mathcal{M} \rangle \leqslant 0$.*
(iii) *Let $\bar{\mathcal{N}} \leqslant 0$, $\bar{\mathcal{M}} \leqslant 0$, and $\langle \mathcal{N}, \mathcal{M} \rangle \leqslant 0$. Then $\overline{\mathcal{N} + \mathcal{M}} \leqslant 0$.*

Proof of Theorem 5.1. Let $\mathcal{N} \in NC$ and $\mathcal{N} = \mathcal{M}_1 + \cdots + \mathcal{M}_k$, with $\mathcal{M}_i \in LC$. According to Lemma 5.3(ii) we have that $\mathcal{M}_i \in BS^{\times *}$ with $\mathcal{M}_i^{\downarrow} \in BS^* \cap LC$, $\langle \mathcal{M}_i^{\downarrow} \rangle \leqslant 0$ in the canonical decomposition:

$$\mathcal{M}_i = \sum_{A \in G_i} \mu_A A + \mathcal{M}_i^{\downarrow}, \; i = 1, ..., k,$$

where G_i is a finite subset of PC.

Denoting $\mathcal{N}' = \mathcal{M}_1^{\downarrow} + \cdots \mathcal{M}_k^{\downarrow}$ and using Theorem 3.4(ii) and Lemma 5.5, we get

$$\mathcal{N}' \in BS^*, \bar{\mathcal{N}}' \leqslant 0, \langle \mathcal{N}' \rangle \leqslant 0 \tag{5.3}$$

and

$$\mathcal{N} = \sum_{A \in G} \mu_A A + \mathcal{N}', \tag{5.4}$$

with $G = G_1 \cup \cdots \cup G_k$.

Our aim is to get a canonical decomposition for $\mathcal{N}$ by changing the representation (5.4) as follows:

(1) If there are $B, D \in G$ such that $\langle B, D \rangle < 0$, (*let* $\mu_B \leqslant \mu_D$) then

$$\mathcal{N} = \sum_{A \in G,\, A \neq B,\, D} \mu_A A + (\mu_D - \mu_B) D + \mathcal{N}'',$$

with $\mathcal{N}'' = \mathcal{N}' + \mu_B(B + D)$.

(2) If there is $B \in G$ such that $\langle B, \mathcal{N}' \rangle = -\mu < 0$, then

$$\mathcal{N} = \sum_{A \in G,\, A \neq B} \mu_A A + (\mu_B - \mu)_+ B + \mathcal{N}'',$$

with $\mathcal{N}'' = \mathcal{N}' + [\mu_B - (\mu_B - \mu)_+] B$. In both cases $\mathcal{N}''$ satisfies the conditions (5.3) due to Lemmas 5.4 and 5.5. It is not hard to see, that after finite number of steps (1) and (or) (2) we will get the canonical decomposition (4.1) for $\mathcal{N}$ with $\mathcal{N}^{\downarrow} \in LC$ and $\langle \mathcal{N}^{\downarrow} \rangle \leqslant 0$. According to Corollary 4.6 this completes the proof.

Using the same arguments we get part (i) of the following

Theorem 5.6. (i) *Suppose that the scheme $\mathcal{N}$ has a decomposition (not necessarily orthogonal)*

$$\mathcal{N} = \sum_{A \in G} \mu_A A + \mathcal{N}', \tag{5.5}$$

where G is a finite subset of PC and $\mathcal{N}' \in BS^*$. Then $\mathcal{N}$ is r-basic and there are coefficients μ'_A, $0 \leqslant \mu'_A \leqslant \mu_A$, $A \in G$ such that the following decomposition is canonical

$$\mathcal{N} = \sum_{A \in G'} \mu'_A A + \mathcal{N}'',$$

where $G' = \{A \in G \colon \mu'_A \neq 0\}$ and $\mathcal{N}'' = \mathcal{N}' + \sum_{A \in G} (\mu_A - \mu'_A) A$.

(ii) *If* $\mathcal{N}_\nu \in BS^{\times *}$, $\nu = 1, ..., k$ *then*

$$\sum_{\nu=1}^{k} \lambda_\nu \mathcal{N}_\nu \in BS^{\times *},$$

where $\lambda_\nu \in Z_+$.

To prove part (ii) it is enough to take the sum of canonical decompositions of $\mathcal{N}_\nu$, $\nu = 1, ... k$ and make use of Theorem 3.4(ii) in order to have a representation for $\mathcal{N}$ of form the (5.5).

The most interesting corollary of Theorem 5.1 is the following exact *numerical analog* of Bézout's theorem:

THEOREM 5.7. *Let* $\mathcal{N}, \mathcal{M}$ *be numerical curves with* $\langle \mathcal{N}, \mathcal{M} \rangle > 0$. *Then* $\mathcal{N}$ *and* $\mathcal{M}$ *have a common prime curve in their canonical decompositions, i.e., there is* $A \in PC$ *such that* $\langle A, \mathcal{N} \rangle > 0$, $\langle A, \mathcal{M} \rangle > 0$.

Proof. Consider the canonical decompositions of $\mathcal{N}, \mathcal{M}$ and assume that $PC_{\mathcal{N}} \cap PC_{\mathcal{M}}$ is empty. Then Theorem 3.3 implies that $\langle C, D \rangle \leqslant 0$ for any schemes C, D belonging to decompositions of $\mathcal{N}, \mathcal{M}$ respectively, provided that one of them is prime. The same for the remaining couple (i.e. $\langle \mathcal{N}^{\downarrow}, \mathcal{M}^{\downarrow} \rangle \leqslant 0$) follows from Theorem 5.1 and Lemma 5.5(i). These inequalities imply $\langle \mathcal{N}, \mathcal{M} \rangle \leqslant 0$, which contradicts the hypothesis of Theorem.

The following Corollary follows from Theorem 5.1 and Corollary 4.5.

COROLLARY 5.8. *Let* $\mathcal{N}$ *be a numerical curve. Then*

(i) $\langle \mathcal{N} \rangle > 0$ *implies* $PC_{\mathcal{N}}$ *is not empty, i.e. there is an* $A \in PC$ *such that* $\langle A, \mathcal{N} \rangle \geqslant 1$,

(ii) $\mathcal{N} \notin LC$ *implies that* $PC_{\mathcal{N}}$ *contains at least one double prime curve, i.e. there is* $A \in PC$ *such that* $\langle A, \mathcal{N} \rangle = \mu_A \geqslant 2$. *In addition here we have* $\deg A \leqslant \deg \mathcal{N}/2$.

We get from this, in particular, the following characterization for those numerical curves which are interpolation schemes.

COROLLARY 5.9. *The interpolation scheme* $\mathcal{N}$ *is a numerical curve if and only if there is a prime curve* $A \in PC$ *with*

$$\langle A, \mathcal{N} \rangle \geqslant 2. \tag{5.6}$$

Proof. It is enough to show that (5.6) implies that $\mathcal{N}$ is a numerical curve. In fact $\mathcal{N}$ can be presented as a sum of two schemes from LC one of which is a prime curve (cf. [P92]). Namely, we have $\mathcal{N} = A + (\mathcal{N} - A)$, since the inequality (5.6) implies that $(\mathcal{N} - A) \in LC$. Indeed, according to (5.2) we have

$$\overline{\mathcal{N} - A} = \overline{A + (\mathcal{N} - A)} - \overline{A} - \langle A, \mathcal{N} - A \rangle = 2 - \langle A, \mathcal{N} \rangle \leqslant 0.$$

COROLLARY 5.10. *An e-Basic scheme is a numerical curve if and only if it is in the class LC.*

Proof. We have from Theorem 3.3 and Remark 4.3 that the canonical decomposition of an *e*-basic scheme $\mathcal{N}$ cannot contain a prime curve i.e. it is trivial: $\mathcal{N} = \mathcal{N}$. Now, it remains to apply Theorem 5.1.

COROLLARY 5.11. *The Conjectures* 1.6 *and* 1.11 *are equivalent*, *i.e. they are both simultaneously valid or not*

Proof. If Conjecture 1.6 is valid then Corollary 5.10 implies that Conjecture 1.11 is valid too. Now let Conjecture 1.11 be true and $\mathcal{N}$ be singular. Then according to Remark 3.2 $\mathcal{N}$ is *r*-basic and we get from Corollary 4.6, that $\mathcal{N} \rightarrow \mathcal{N}^{\downarrow} \in BS^*$. Making use of Remark 2.3 we obtain that $\mathcal{N}^{\downarrow}$ is singular and therefore it belongs to LC by Corollary 2.4(i) and Conjecture 1.11. Now it follows from the canonical decomposition of $\mathcal{N}$ that it is numerical curve. And hence Conjecture 1.6 is true.

Remark. Some applications of numerical curves in Algebraic Geometry given in the paper [GHS95].

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